§1.9 The Matrix of a Linear Transformation

Last time we saw that matrix transformations were linear transformations. Today we'll see the opposite holds as well, every linear transformation can be expressed as a matrix transformation.

In other words, if $T: \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation, there is an $m \times n$ matrix A such that $T(x) = Ax$.

Question: Given T, can we find A?

Defn

The $n \times n$ matrix $I_n = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \ddots & \vdots \\ \vdots & 0 & \ddots & \vdots \\ 0 & 0 & \cdots 0 & 1 \end{bmatrix}$ with 1's on the diagonal and 0's everywhere else is called the identity matrix of $\mathbb{R}^n$.

We call it this since it corresponds to the identity transformation

$$T: \mathbb{R}^n \longrightarrow \mathbb{R}^n \qquad \text{since } Ax = x$$
$$x \longmapsto x \qquad \text{for } x \text{ in } \mathbb{R}^n$$

We denote the columns of I_n by $e_1, \ldots, e_n$

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \cdots\cdots, \quad e_n = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

<u>Example</u>

In $\mathbb{R}^2$ $e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Suppose T is the transformation where $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ and

$$T(e_1) = \begin{bmatrix} 3 \\ -2 \\ 16 \end{bmatrix} \qquad T(e_2) = \begin{bmatrix} 2 \\ 1 \\ 23 \end{bmatrix}$$

what is the matrix A such that $T(x) = Ax$?

$$T(x) = T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = T\Big(x_1 \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{e_1} + x_2 \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{e_2}\Big)$$

$$= x_1 T(e_1) + x_2 T(e_2)$$

$$= x_1 \begin{bmatrix} 3 \\ -2 \\ 16 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 1 \\ 23 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 2 \\ -2 & 1 \\ 16 & 23 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\boxed{A = \begin{bmatrix} 3 & 2 \\ -2 & 1 \\ 16 & 23 \end{bmatrix}}$$ since $T(x) = Ax$

Theorem

Let $T: \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. The matrix $A = \left[T(e_1) \middle| \cdots \middle| T(e_n) \right]$ is the unique $m \times n$ matrix such that $T(x) = Ax$.

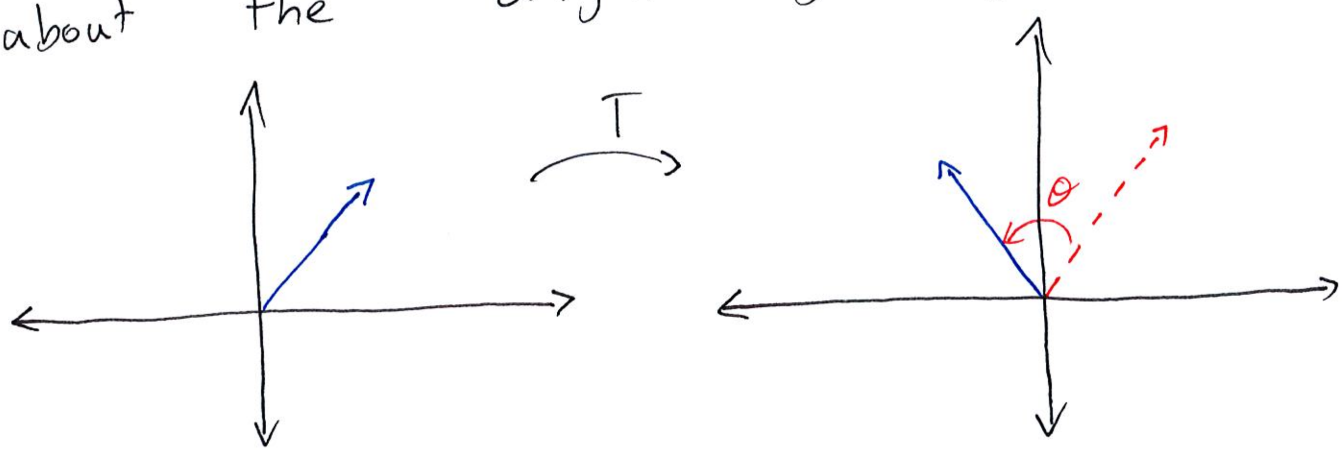
Defn

This matrix A is called the <u>standard matrix for the linear transformation</u> T.

Notice this allows us to describe complicated transformations by considering what it does to a short list of vectors.

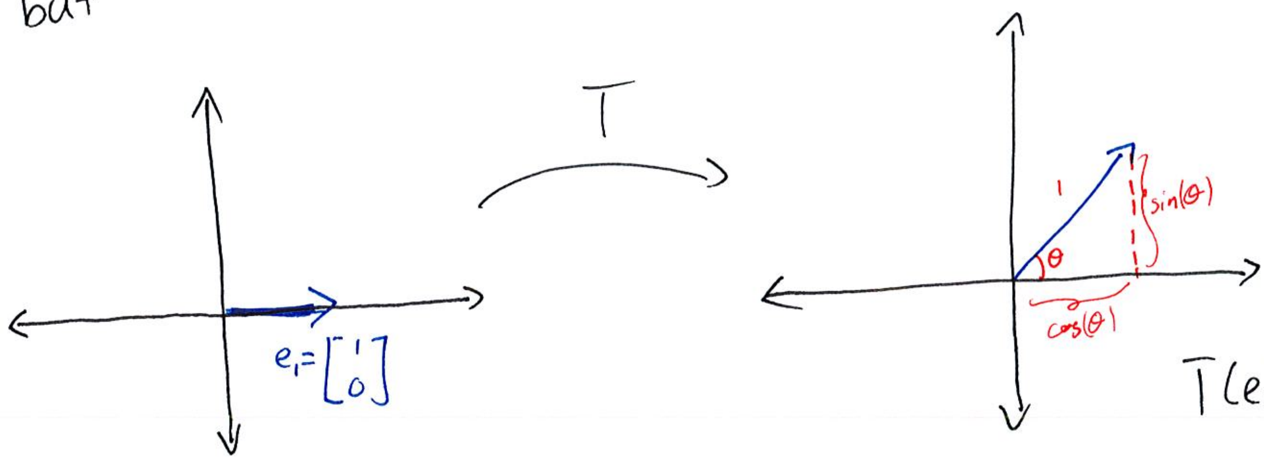
Example

Let T be the linear transformation $T: \mathbb{R}^2 \to \mathbb{R}^2$ that rotates vectors counter-clockwise about the origin by angle θ.

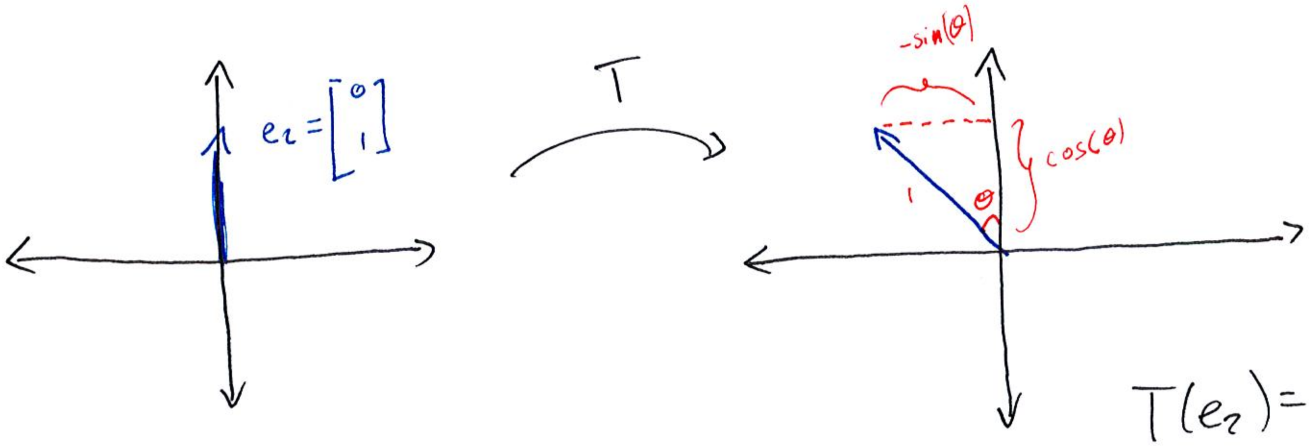


For any $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ it's not clear what $T(x)$ is

but



$$T(e_1) = \begin{bmatrix} \cos(\theta) \\ \sin(\theta) \end{bmatrix}$$



$$T(e_2) = \begin{bmatrix} -\sin(\theta) \\ \cos(\theta) \end{bmatrix}$$

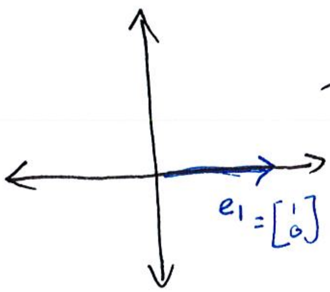
Thus $T(x) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} x$

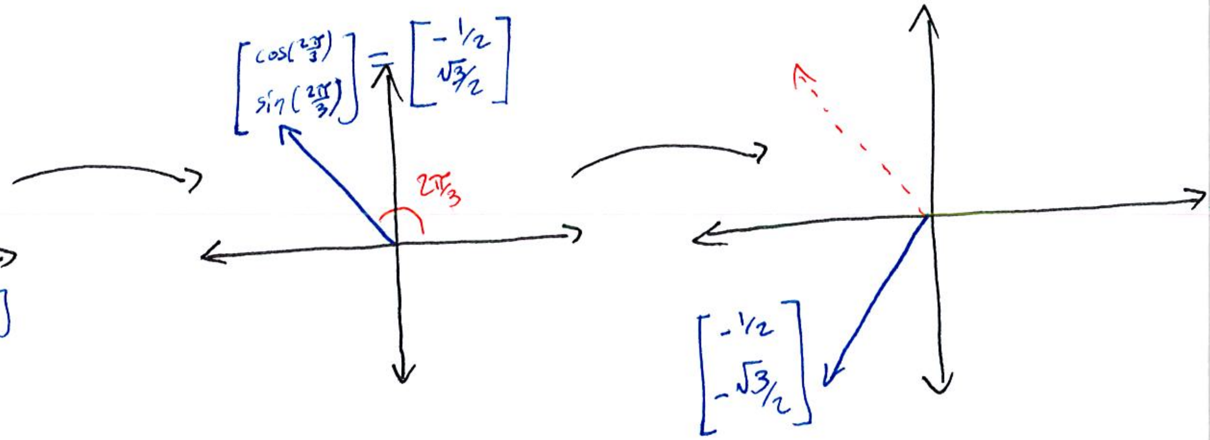
This matrix is the <u>rotation matrix</u>.

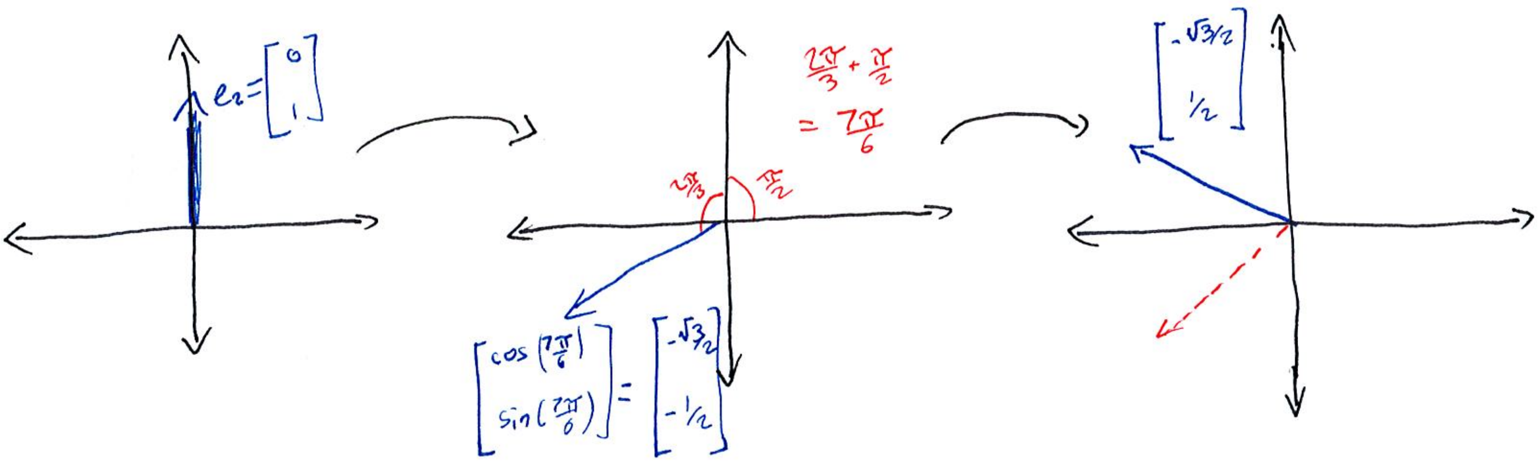
Example

Let $T:\mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation that first rotates points by $\frac{2\pi}{3}$ radians and then reflects points across the horizontal x_1-axis

what is A?







Thus

$$T(x) = \left[T(e_1) \mid T(e_2) \right] x$$

$$= \begin{bmatrix} -1/2 & -\sqrt{3}/2 \\ -\sqrt{3}/2 & 1/2 \end{bmatrix} x$$

Thus $A = \begin{bmatrix} -1/2 & -\sqrt{3}/2 \\ -\sqrt{3}/2 & 1/2 \end{bmatrix}$ is the matrix of the transformation

Defn

Let $T: \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation

1) T is said to be onto $\mathbb{R}^m$ if every b in $\mathbb{R}^m$ is the image of at least one x of $\mathbb{R}^n$.

I.e. given any b in $\mathbb{R}^m$, there's an x of $\mathbb{R}^n$ such that $T(x) = b$

2) T is one-to-one if each b in $\mathbb{R}^m$ is the image of at most one x in $\mathbb{R}^n$.

I.e. if u, v in $\mathbb{R}^n$ such that $T(u) = b = T(v)$, then $u = v$.

Theorem: Let $T: \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation

a) T is one-to-one if and only if the equation $T(x) = 0$ has only the trivial solution.

b) T is one-to-one if and only if the columns of A are linearly independent where $T(x) = Ax$.

c) T is onto if and only if the columns of A span $\mathbb{R}^m$.